The distance b/w a point 'p' $\in X$ and A is denoted by $d(p, A)$ and is defined by $d(p,A) = \inf\{d(p,a) : a \in A\}$

if $p \in A$ then $d(p, A) = 0$

The distance b/w two non-empty subsets A and B of a metric space X is defined by $d(A,B) = \inf\{d(x,y) : x \in A, y \in B\}$

The set $d(x,y)$ is bounded below by zero i.e., $d(x,y) \geq 0$

<u>Theorem</u> – Let A be a non-empty set in a metric space (X, d) and let (x, y) be any two points on X. Then

$$|d(x,A) - d(y,A)| \leq d(x,y)$$

<u>Proof</u>: We know

$$\begin{aligned} d(x,A) &= \inf\{d(x,z) : z \in A\} \\ &\leq d(x,z) \ \forall z \in A \\ &\leq d(x,y) + d(y,z) \ \forall z \in A \end{aligned}$$

$$\Rightarrow d(x,A) - d(x,y) \leq d(y,z) \ \forall z \in A \quad (1)$$

$\cancel{d(x,A) - d(x,y) \leq \inf\{d(y,z)\} \ z \in A}$

Similarly,

$$d(y,A) - d(x,A) \leq d(y,x) = d(x,y) \quad (2)$$

Since d is symmetric

From eqn ① & ② we get

$$|d(x,A) - d(y,A)| \leq d(x,y)$$

Proved

Ques Let (X,d) be a metric space and A, B are subsets of X. Prove that $\delta(A \cup B) \leq \delta(A) + \delta(B) + d(A,B)$. where $\delta(A) \equiv$ diameter of the set A.

Soln Let $x, y \in A \cup B$ and it be arbitrary. Obviously either $x, y \in A$ or $x, y \in B$ or $x \in A, y \in B$ and $x \in B, y \in A$

if $x, y \in A$ then clearly $\delta(A) \geq d(x,y)$ — ①

if $x, y \in B$ then clearly $\delta(B) \geq d(x,y)$ — ②

if $x \in A$ and $y \in B$ then we get

$$d(x,y) \leq d(x,a) + d(a,y) \text{ — (by M4)}$$
$$\leq d(x,a) + d(a,b) + d(b,y) \text{ — (by M4)}$$
$$\leq \delta(A) + d(a,b) + \delta(B) \text{ — ③}$$

Similarly if $x \in B, y \in A$ we get

$$d(x,y) \leq \delta(B) + d(a,b) + \delta(A) \text{ — ④}$$

From Eqn ① & ② we get

$$d(x,y) \leq \delta(A) + d(a,b) + \delta(B)$$

$$\Rightarrow \delta(A \cup B) \leq \delta(A) + d(a,b) + \delta(B)$$

$\because$ it is true for any $a \in A, b \in B$

we get $\delta(A \cup B) \leq \delta(A) + d(A,B) + \delta(B)$

Proved

Spheres in R^n

If $a \in R^n$ and 'r' is any positive real no. then $S(a,r) = \{x \in R^n : d(x,a) < r\}$ and is called the set on an open sphere with centre 'a' and radius 'r'.
if $S(a,r) = \{x \in R^n : d(x,a) \leq r\}$ then it is called a closed sphere with centre 'a' and radius 'r'.

Spherical neighbourhood of a point

Any open sphere with centre 'a' is called spherical neighbourhood of a point 'a'. Thus $S(a,r)$ is a spherical neighbourhood of the point 'a' $\forall$ $r>0$.

Rectangular neighbourhood

Let $a = (a_1, a_2, \dots, a_n)$ and $b = (b_1, b_2, \dots, b_n)$ be two points in R^n such that

$$a_1 < b_1,\ a_2 < b_2 \dots,\ a_n < b_n$$

then the set of points

$$x = (x_1, x_2, \dots, x_n) \text{ where } a_1 < x_1 < b_1$$

$a_2 < x_2 < b_2, \dots, a_n < x_n < b_n$.

is called an open rectangle in R^n and is denoted by $]a,b[$

or $]a_1, b_1 : a_2, b_2 : \dots : a_n, b_n[$

Again the set of points x such that
$a_1 \le x_1 \le b_1, a_2 \le x_2 \le b_2, \dots, a_n \le x_n \le b_n$
is called closed rectangle in R^n and is denoted by $[a, b]$
or $[a_1, b_1 : a_2 ; b_2 : \dots : a_n, b_n]$

Any open rectangle containing a point is called rectangular neighbourhood of that ~~poi~~ point.